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FUZZY OPTIMAL CONTROL PROBLEM WITH NON-LINEAR FUNCTIONAL

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In the present work, first the space of fuzzy numbers is constructed and a scalar product is introduced. The derivative of fuzzy function in this space is defined. Using this approach the method is proposed to investigate the fuzzy optimal control problem and offer an algorithm for its numerical solution.

Any situation that requires decision making, as a rule, contains a great amount of indeterminacy. The control problems containing indeterminacies have been studied for example, in the papers [1, 2]. To investigate fuzzy optimal control problem, first, one has to introduce the definition of the derivative of the fuzzy function [2]. This definition must allow one to investigate optimal control problems both theoretically and numerically.

Let's define by F the class of convex normal fuzzy numbers. Then for any $a \in F$ the set of α-cut of fuzzy number a the interval $a^\alpha = [L_a(\alpha), R_a(\alpha)]$, $\alpha \in [0,1]$, is defined ([3]). Let $a \in F$, $b \in F$ and $a^\alpha = [L_a(\alpha), R_a(\alpha)]$, $b^\alpha = [L_b(\alpha), R_b(\alpha)]$. Then α-cut of fuzzy number $a+b$ and $ka, k \ge 0$, definesas $a^\alpha + b^\alpha = [L_a(\alpha) + L_b(\alpha), R_a(\alpha) + R_b(\alpha)]$ and $ka^\alpha = [kL_a(\alpha), kR_a(\alpha)]$, respectively.

Note that F is not a linear space (the operation of subtraction is not defined in F).

We consider the set of pairs $(a,b) \in F \times F$ and define the operation of addition, multiplication and equivalency as

$$\begin{aligned}
&(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),\\
&k \cdot (a,b) = (ka,\ kb), \quad k \ge 0,\\
&(-1) \cdot (a,b) = (b,a),\\
&(a_1, a_2) \approx (b_1, b_2) \iff a_1 + b_2 = a_2 + b_1.
\end{aligned} \tag{1}$$

As zero element of this space is taken the pair $(0,0)$, i.e. the set of elements $(a,\ a)$, $a \in F$.

The set of all pairs $(a,b) \in F \times F$ forms a structure of a linear space. Let

$$x = (a_1, a_2) \in F \times F, \;\; y = (b_1, b_2) \in F \times F \, .$$

Then

$$a_i^{\alpha} = [L_{a_i}(\alpha), R_{a_i}(\alpha)], \quad b_i^{\alpha} = [L_{b_i}(\alpha), R_{b_i}(\alpha)], \; \alpha \in [0,1] \, .$$

For any $x, y \in F \times F$ define the scalar product as

$$x \circ y = \int_0^1 [(L_{a_1}(\alpha) - L_{a_2}(\alpha))(L_{b_1}(\alpha) - L_{b_2}(\alpha)) +$$

$$+ (R_{a_1}(\alpha) - R_{a_2}(\alpha))(R_{b_1}(\alpha) - R_{b_2}(\alpha)(\alpha)] d\alpha \, . \qquad (2)$$

It may be shown that this definition satisfies all the requirements of the scalar product. We denote this space by LF . The norm in this space is defined as

$$\| x \|^2 = \int_0^1 [(L_{a_1}(\alpha) - L_{a_2}(\alpha))^2 + (R_{a_1}(\alpha) - R_{a_2}(\alpha))^2] d\alpha \, , \qquad (3)$$

We define the distance between two fuzzy numbers $a \in F$ and $b \in F$ as

$$\rho(a,b) = \| x - y \| \, , \qquad (4)$$

where $x = (a, 0), y = (b, 0)$. We also will use the norm

$$\| x \|_{CF} = \max_{\alpha \in [0,1]} \left(\left| L_{a_1}(\alpha) - L_{a_2}(\alpha) \right| + \left| R_{a_1}(\alpha) - R_{a_2}(\alpha) \right| \right) .$$

Now, let's consider fuzzy function $f(t) \in F$ for each $t \in [t_0, t_1]$ and define a derivative of the function $f(t)$.

For any $\alpha \in [0,1]$,

$$f_{\alpha}(t) = [L_{f(t)}(\alpha), R_{f(t)}(\alpha)], \alpha \in [0,1] \qquad (5)$$

is called α -cut of the function $f(t)$.

Definition. Let there exists such $\varphi(t), \psi(t)$, that

$$\lim_{\Delta t \to 0} \frac{(f(t + \Delta t), 0) - (f(t), 0)}{\Delta t} = (\varphi(t), \psi(t)) \, . \qquad (6)$$

Then the pair $(\varphi(t), \psi(t)) \in F \times F$ is a derivative of the function $f(t)$ at the point $t \in (t_0, t_1)$. This definition may be written in the following form

$$\lim_{\Delta t \to 0} \frac{(f_{\alpha}(t + \Delta t), 0) - (f_{\alpha}(t), 0)}{\Delta t} = (\varphi_{\alpha}(t), \psi_{\alpha}(t)) \, ,$$

where $\varphi_{\alpha}(t), \psi_{\alpha}(t)$ are α -cut for the functions $\varphi(t), \psi(t)$.

It is shown that, if $L_{f(t)}(\alpha), R_{f(t)}(\alpha)$ is continuous differentiable relatively t , then

$f(t)$ is differentiable. Each function $f(t)$ may be considered as an element $(f(t),0)$ from $F \times F$. Then

$$(f_1(t) \pm f_2(t))' = f_1'(t) \pm f_2'(t).$$

Now, let $f(t)$ be a pair of fuzzy functions, i.e.

$$f(t) = (f_1(t), f_2(t), \ \forall t \in (t_0, t_1).$$

From relation

$$f(t) = (f_1(t),0) + (0, f_2(t)) = f_1(t),0) - (f_2(t),0),$$

we see, that the derivative of the function $f(t)$ also is a pair from $F \times F$.
Denote

$$L'_{f(t)}(\alpha) = \frac{\partial}{\partial t} L_{f(t)}(\alpha),$$

$$R'_{f(t)}(\alpha) = \frac{\partial}{\partial t} R_{f(t)}(\alpha).$$

If $L'_{f(t)}(\alpha) \le R'_{f(t)}(\alpha)$, then

$$f'(t) = ([L'_{f(t)}(\alpha), R'_{f(t)}(\alpha)],0).$$

Otherwise, i.e. $L'_{f(t)}(\alpha) \ge R'_{f(t)}(\alpha)$, we obtain

$$f'(t) = (0,[-L'_{f(t)}(\alpha), -R'_{f(t)}(\alpha)]).$$

For any $\eta = \eta(t) \in F \times F$, which $\eta'(t) \in F \times F$, $t \in [t_0, t_1]$, consider the scalar product $f'(t) \circ \eta(t)$ defined by the formulae (2). It can be shown that

$$\int_t^T f'(t) \circ \eta(t) dt = f(t) \circ \eta(t) /_t^T - \int_t^T f(t) \circ \eta'(t) dt, \ \forall t, T \in [t_0, t_1]. \qquad (7)$$

One may show that this derivative satisfies the "necessary natural" conditions.
Example. Let $f(t)$ be fuzzy function whose α - cut is defined as follows:

$$f_\alpha(t) = [\alpha - 1 + t, 1 - \alpha + t].$$

Then it is not difficult to show that

$$f'_\alpha(t) = (\varphi_\alpha(t), \psi_\alpha(t)),$$

where

$$\varphi_\alpha(t) = [\alpha, 2 - \alpha], \ \ \psi_\alpha(t) = [\alpha - 1, 1 - \alpha],$$

Now, consider the following differential equation

$$x'(t) = A(t)x(t) + B(t)v(t), \ \ 0 < t \le T, \qquad (8)$$

with the initial condition

$$x(0) = x_0, \qquad (9)$$

where $v(t) \in V_0$ for each $t \in [0, T]$; V_0 is a convex subset of the F; $A(t), B(t)$ are the known functions; $x_0 \in F$ is a given fuzzy number. Let

$x(t)=(x_1(t),x_2(t))\in F\times F$ and $x'(t)=(\varphi_1(t),\varphi_2(t)\in F\times F$. We will understand equality (8) as equality in space LF

$$(\varphi_1(t),\varphi_2(t))=A(t)(x_1(t),x_2(t))+B(t)(v(t),0),\ 0<t\le T. \qquad (10)$$

Denote $V=\{v=v(t)\in \overset{\circ}{V_0},\ \ \forall t\in[0,T],\ \|v(t)\|\in L_2(0,T)\}$. Here we denote

$$\|v(t)\|=\|(v(t),0)\|_{LF}$$

Let for any $t\in[0,T],\ x\in F\times F, v\in F$ $\quad f(t,x,v)\in R, g(x)\in R$ and $|f(t,x,v)|\le C(\|x\|^2+\|v\|^2)$, where C is a positive constant.

Consider the following optimization problem:

To find $v(t)\in V$ from condition of maximum of the following functional

$$\int_0^T f(t,x(t),v(t)dt+g(x(T))\to \min \qquad (11)$$

where $x=x(t)$ is found as a solution of the problem (8)-(9) corrosponding $v=v(t)$. Let $A(t),\ B(t)$ be continous functions.

Lemma. If $x_0=(x_0^{(1)},x_0^{(2)})\in F\times F$. Then for any $v=v(t)\in V$ there exists the unique solution $x(t)\in F\times F$ of the problem (8), (9) and $\|x(t)\|\le const, \forall t\in[0,T]$.

Proof. Let $v=v(t)\in V$ be any fuzzy number with α-cut $v_\alpha(t)=[L_{v(t)}(\alpha),R_{v(t)}(\alpha)]$, $x_0^\alpha=[L_{x_0}(\alpha),R_{x_0}(\alpha)]$ and $B(t)=B_1(t)-B_2(t)$, $B_i(t)\ge 0,\ i=1,2.$ Taking

$$L_{x(t)}^{(i)}(\alpha)=p(t)\left[L_{x_0^{(1)}}(\alpha)+\int_0^t B_i(\tau)\exp\left(-\int_0^\tau A(s)ds\right)L_{v(\tau)}(\alpha)d\tau\right],$$

$$R_{x(t)}^{(i)}(\alpha)=p(t)\left[R_{x_0^{(2)}}(\alpha)+\int_0^t B_i(\tau)\exp\left(-\int_0^\tau A(s)ds\right)R_{v(\tau)}(\alpha)d\tau\right], \qquad (12)$$

consider the pair $x(t)\in(x^{(1)}(t),x^{(2)}(t))$ with α-cuts $x_\alpha^{(i)}(t)=[L_{x(t)}^{(i)}(\alpha),R_{x(t)}^{(i)}(\alpha)], i=1,2$, $\alpha\in[0,1]$. here $p(t)=\exp\left(\int_0^t A(\tau)d\tau\right)$ Take into account that $L_{x_0^{(i)}}(\alpha)\le R_{x_0^{(i)}}(\alpha),\ L_{v(\tau)}(\alpha)\le R_{v(\tau)}(\alpha)$, it is not difficult to show that $L_{x(t)}^{(i)}\le R_{x(t)}^{(i)}$, for any $\alpha\in[0,1]$. Here Checking this we will see, that $x(t)\in F\times F$, $t\in[0,T]$, is a solution of the problem (8), (9). The proof of the uniqueness of the problem (8), (9) is not difficult. Take into account that $\|v(t)\|\in L_2(0,T)$ from (12) we get $\|x(t)\|\le const, \forall t\in[0,T]$.

The Lemma is proved.

Remark. If $B(t)\ge 0,\ t\in[0,T]$ and $x_0\in F$, then for any $v=v(t)\in V$ there exists the unique solution $x(t) \in F$ of the problem (8), (9).

Let $\psi = \psi(t)$ be a solition of the problem

$$\psi'(t) = -A(t)\psi(t) + f_x(t, x(t), v(t)), \tag{13}$$

$$\psi(t) = -g_x(x(T). \tag{14}$$

The problem (13), (14) is called adjoint problem for (8), (9).

Let for any $t \in [0, T[$, the functional $x \to f(t, x, v)$ be differentable on LF, i.e.

$$|f(t, x+h, v) - f(t, x, v) - f_x(t, x, v) \circ h| \le L(h)\|h\|,\ h \in LF,\ \forall t \in [0, T].$$

Here $L(h) \to 0,$ when $\|h\| \to 0$.

The following theorem is proved.

Theorem 1. Let $\{v^*(t), x^*(t)\}$ be an optimal pair for the problem (8)-(10) . Then for each $\overset{\circ}{\forall} t \in (0, T)$ the following relation is fulfilled

$$\begin{aligned} &B(t)v^*(t) \circ \psi^*(t) - f(t, x^*(t), v^*(t)) = \\ &= \max_{v \in V_0}[B(t)v \circ \psi^*(t) - f(t, x^*(t), v)], \end{aligned} \tag{15}$$

here $\psi^* = \psi^*(t)$ is the solution of the problem (13)-(14) corrosponding $v^* = v^*(t)$.

Let for any $t \in [0, T[$, the functional $v \to f(t, x, v)$ be differentable on LF.

Difinition. The functional $J(v)$ is called differentiable in $v \in V$, if there exist the limit

$$\lim_{\|h\| \to 0} \frac{J(v+h) - J(v) - J'(v) \circ h}{\|h\|} = 0 \cdot \tag{16}$$

Theorem 2. The functional $J(v)$ differentiable in $v \in V$ and

$$J'(v) = B(t)\psi - f_v(t, x, v). \tag{17}$$

Theorem 3. If the element $v^* \in V$ gives minimum to functional $J(v)$ under condition (8),(9), then

$$\int_0^T \left[f_v(t, x^*(t), v^*(t)) - B(t)\psi^*(t)\right] \circ (v(t) - v^*(t))dt \ge 0,\ \forall v \in V. \tag{18}$$

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QEYRİ-XƏTTİ FUNKSİONAL İLƏ FAZZİ OPTİMAL İDARƏETMƏ MƏSƏLƏSİ

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XÜLASƏ

İşdə fazzi ədədlərin fəzası qurulur və skalyar hasil təyin edilir. Bu fəzada fazzi funksiyanın törəməsi təyin edilir. Optimal idarəetmənin fazzi məsələsini tədqiq etmək üçün metod və ədədi həll üçün alqoritm təklif edilir.

ФАЗЗИ ЗАДАЧИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ С НЕЛИНЕЙНЫМ ФУНКЦИОНАЛОМ

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РЕЗЮМЕ

В работе построено пространство фаззи чисел и введено скалярное произведение. В этом пространстве определена производная фаззи функций. Предложен метод для исследования фаззи задачи оптимального управления и алгоритм для численного решения.